



Security of Keyed Hashing Based on a Public Permutation

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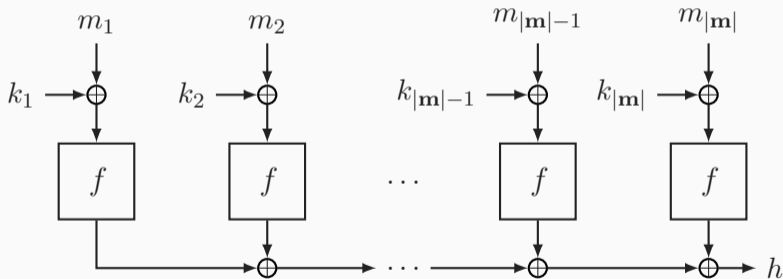
Permutation-Based Crypto

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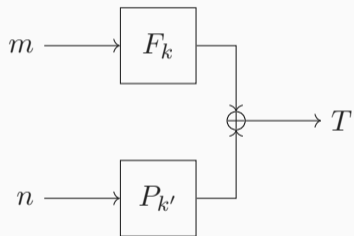
The Parallelization of a Public Permutation

- Parallel keyed hashing with an underlying public permutation
- $\max_{a, \Delta \in G} \text{DP}_f(a, \Delta)$ - Δ universal and $\max_{a \in G} \sum_{t \in G} \text{DP}_f^2(a, t)$ -universal

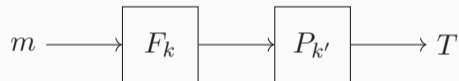


Motivation

- Keyed hash functions take arbitrary length inputs and output a fixed length digest
- We study their security when used in Wegman-Carter(-Shoup) and protected hash



Wegman-Carter(-Shoup)



Protected Hash

- Security against forgery of the tuple (m, n, T)
- The attacker has to come up with a tuple (m^*, n^*, T^*) :

$$T = F_k(m) + P_{k'}(n)$$

$$T^* = F_k(m^*) + P_{k'}(n^*)$$

$$T - T^* = F_k(m) - F_k(m^*) + P_{k'}(n) - P_{k'}(n^*)$$

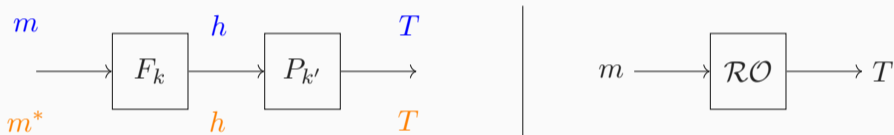
$$F_k(m) - F_k(m^*) = T - T^* - P_{k'}(n) + P_{k'}(n^*)$$

- ϵ - Δ universality [Sti95] of F is defined over all distinct pairs of messages m, m^* :

$$Pr[F_k(m) - F_k(m^*) = \Delta] \leq \epsilon$$

Security of Protected Hash

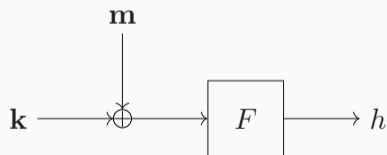
- The security is defined as a distinguishing advantage
- Assuming that $P_{k'}$ is PRP-secure and $m \neq m^*$:



- ϵ -universality [Sti95] of F is defined over all distinct pairs of messages m, m^* :

$$\Pr[F_k(m) = F_k(m^*)] \leq \epsilon$$

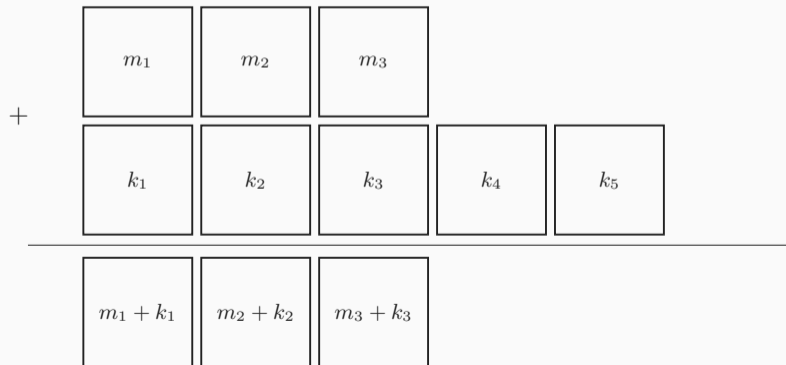
- A Key-Then Hash function $F: G^\kappa \times BS(G, \kappa) \rightarrow G$ with:
 - Strings of length 1 to κ over the group $\langle G, + \rangle$: $BS(G, \kappa) = \bigcup_{i=1}^{\kappa} G^i$
 - Key Space: G^κ
 - Output Space: G
- Where $F_{\mathbf{k}}(\mathbf{m}) = F(\mathbf{k} + \mathbf{m})$:



Addition of two strings

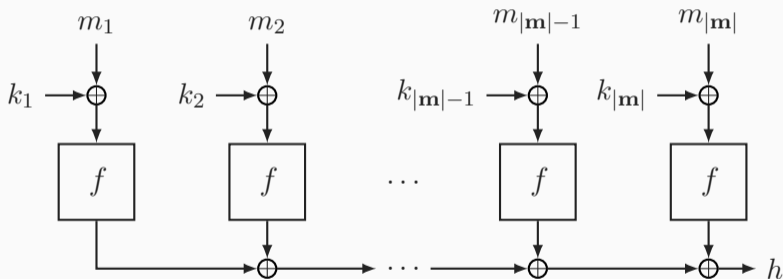
- We define the addition of any two strings $\mathbf{m}, \mathbf{m}^* \in BS(G, \kappa)$ with $|\mathbf{m}| \leq |\mathbf{m}^*|$ as:

$$\mathbf{m}' = m_1 + m_1^*, m_2 + m_2^*, m_3 + m_3^*, \dots, m_{|\mathbf{m}|} + m_{|\mathbf{m}|}^*$$



The Parallelization of a Public Permutation

- $\text{Parallel}[f]$ builds a key-then-hash function using a public permutation f



Differential Probability Over Fixed-Length Permutations

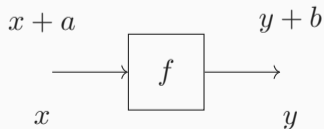
Given fixed-length permutation $f: G \rightarrow G$

- Input difference $a \in G$ propagates to the output difference $b \in G$ through f if

$$f(x + a) - f(x) = b$$

- The pair (a, b) is called a differential over f and happens with probability:

$$\text{DP}_f(a, b) = \frac{\#\{x \in G \mid f(x + a) - f(x) = b\}}{\#G}$$

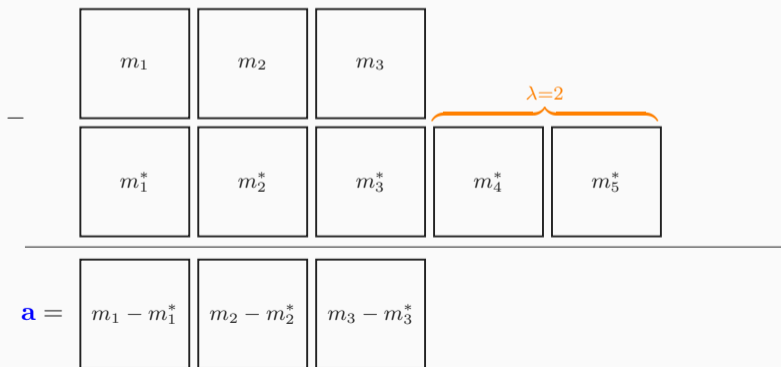


A Difference Between Two Strings

- We define the difference of any two strings \mathbf{m}, \mathbf{m}^* with $|\mathbf{m}| \leq |\mathbf{m}^*|$ as:

$$\mathbf{a} = m_1 - m_1^*, m_2 - m_2^*, m_3 - m_3^*, \dots, m_{|\mathbf{m}|} - m_{|\mathbf{m}|}^*$$

$$\lambda = |\mathbf{m}^*| - |\mathbf{m}|$$



- We are interested in proving bounds on the universality and Δ -universality of Parallel[f]
 - Input difference (\mathbf{a}, λ) leading to a Δ output difference through Parallel[f]:

$$\sum_{i=1}^{|\mathbf{m}|} f(m_i + k_i) - \sum_{j=1}^{|\mathbf{m}^*|} f(m_j^* + k_j) = \Delta$$

- Assuming $|\mathbf{m}| \leq |\mathbf{m}^*|$

$$\sum_{i=1}^{|\mathbf{m}|} f(m_i + k_i) - f(m_i^* + k_i) - \sum_{j=|\mathbf{m}|+1}^{|\mathbf{m}^*|} f(m_j^* + k_j) = \Delta$$

$$\sum_{i=1}^{|\mathbf{m}|} f(m_i + k_i) - f(m_i^* + k_i) - \sum_{j=|\mathbf{m}|+1}^{|\mathbf{m}^*|} f(m_j^* + k_j) = \Delta$$

- $\Pr[f(m_i + k_i) - f(m_i^* + k_i) = b_i]$ with $a_i = m_i - m_i^*$ is given by $\text{DP}_f(a_i, b_i)$
- $\Pr[f(m_j^* + k_j) = x_j]$ is uniform and is equal to $\frac{1}{\#G}$

$$\sum_{i=1}^{|\mathbf{m}|} \underbrace{f(m_i + k_i) - f(m_i^* + k_i)}_{b_i} - \sum_{j=|\mathbf{m}|+1}^{|\mathbf{m}^*|} \underbrace{f(m_j^* + k_j)}_{x_j} = \Delta$$

Assuming $a_i = m_i - m_i^*$:

- b_i can be seen as a stochastic variable with probability mass function

$$\text{DP}_{a_i}(b_i) = \text{DP}_f(a_i, b_i)$$

- x_j can be seen as a stochastic variable with probability mass function

$$U(x_j) = \frac{1}{\#G}$$

- The PMF of a stochastic variable $z = x + y$ is given by

$$g_z(v) = g_x * g_y(v) = \sum_t g_x(t)g_y(v - t)$$

- The resulting convolution is bound by the PMFs of the original two variables

$$\max_v g_z(v) \leq \min \left\{ \max_v g_x(v), \max_v g_y(v) \right\}$$

$$\sum_{i=1}^{|\mathbf{m}|} \underbrace{f(m_i + k_i) - f(m_i^* + k_i)}_{b_i} - \sum_{j=|\mathbf{m}|+1}^{|\mathbf{m}^*|} \underbrace{f(m_j^* + k_j)}_{x_j} = \Delta$$

- The output difference of Parallel[f] is a stochastic variable
- It is the variable resulting from the sum of all b_i and x_j

$$DP_{a_1} * DP_{a_2} * \dots * DP_{a_{|\mathbf{m}|}} * U^{(|\mathbf{m}^*| - |\mathbf{m}|)}(\Delta)$$

- We now define a notion of differentials over Parallel[f] and their probability:

$$DP_{\text{Par}}(\mathbf{a}, \lambda, \Delta) = DP_{a_1} * \dots * DP_{a_{|\mathbf{a}|}} * U^{(\lambda)}(\Delta)$$

- We directly get a bound on the Δ universality

$$\max_{(\mathbf{a}, \lambda, \Delta)} DP_{\text{Par}}(\mathbf{a}, \lambda, \Delta) \leq \max \left\{ \max_{(a, \Delta)} DP_f(a, \Delta), \frac{1}{\#G} \right\} = \max_{(a, \Delta)} DP_f(a, \Delta)$$

- We now prove an upper bound on the universality of Parallel[f]
 - An output difference 0 is not possible with single-block strings
 - From our definition of differentials over Parallel[f] we know

$$\max_{(\mathbf{a}, 0, \Delta)} \text{DP}_{\text{Par}}(\mathbf{a}, 0, \Delta) \geq \max_{(\mathbf{a}, 0, 0)} \text{DP}_{\text{Par}}(\mathbf{a}, 0, 0)$$

- We also know that $\max_{(\mathbf{a}, 0, \Delta)} \text{DP}_{\text{Par}}(\mathbf{a}, 0, \Delta)$ is maximized with $\mathbf{a} \in G^2$
 - Hence we must find

$$\max_{(\mathbf{a} \in G^2, \Delta)} \sum_t \text{DP}_f(a_1, t) \text{DP}_f(a_2, \Delta - t)$$

$$\max_{\mathbf{a} \in G^2, \Delta} \sum_t \text{DP}_f(a_1, t) \text{DP}_f(a_2, \Delta - t)$$

- This can be seen as the inner product of two vectors indexed by t
- The inner product of two vectors is upper bound by the product of their norm (Cauchy-Schwarz)

$$|\langle \mathbf{a}, \mathbf{b} \rangle| \leq \|\mathbf{a}\| \|\mathbf{b}\|$$

$$|\langle \mathbf{a}, \mathbf{b} \rangle| \leq \|\mathbf{a}\| \|\mathbf{b}\| \leq \max \{ \|\mathbf{a}\|^2, \|\mathbf{b}\|^2 \}$$

- Hence we get the following bound

$$\begin{aligned} \sum_t \text{DP}_f(a_1, t) \text{DP}_f(a_2, \Delta - t) &\leq \max \left\{ \sum_t \text{DP}_f^2(a_1, t), \sum_t \text{DP}_f^2(a_2, \Delta - t) \right\} \\ &= \max \left\{ \sum_t \text{DP}_f^2(a_1, t), \sum_t \text{DP}_f^2(a_2, t) \right\} \end{aligned}$$

$$\sum_t \text{DP}_f(a_1, t) \text{DP}_f(a_2, \Delta - t) \leq \max \left\{ \sum_t \text{DP}_f^2(a_1, t), \sum_t \text{DP}_f^2(a_2, t) \right\}$$

- We have proven the following upper bound on the universality

$$\max_{\mathbf{a} \in G^2} \text{DP}_{\text{Par}}(\mathbf{a}, 0, 0) \leq \max_a \sum_t \text{DP}^2(a, t)$$

- This bound is tight when we take $a_2 = -a_1$

Parallelization of a Public Permutation

- We have shown a parallel key-then-hash function built on a public permutation
- It is $\max_{a, \Delta} \text{DP}(a, \Delta)$ - Δ universal and $\max_a \sum_t \text{DP}_f(a, t)^2$ -universal
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More In The Paper

- We define the probability of differential over key-then-hash functions
- We show an analysis to a serial key-then-hash
- We apply these results XOODOO[3] and XOODOO[4]

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Thank you for your attention!

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